§3 Complex Numbers

§3.1 Introduction

What are complex numbers? Before we define this specific type of number, which you will soon learn may be verbal irony, there is something else with priority. Think of it like a building block.

Definition 3.1. The imaginary number \( i \) is the value which satisfies the equation \( x^2 = -1 \).

Why is this referred to as imaginary? It is simply because no real number, that is, a number including any type of decimal, positive or negative, satisfies this. When a real number is squared, it must be positive. Thus, we refer to \( i \) as imaginary, which Oxford defines as ”existing only in the imagination”. However, it may come to you soon that these imaginary concepts could have very real impacts in reality.

We should be well-equipped now for the definition of the complex number.

Definition 3.2. A complex number is a number which can be expressed as

\[
a + bi
\]

for real numbers \( a \) and \( b \) and imaginary unit \( i \).

It’s pretty obvious that \( i \) itself can be classified as a complex number. Why? Well, \( i = 0 + 1i \), and \( 0 \) and \( 1 \) are certainly real numbers that fit for \( a \) and \( b \). Under the same umbrella, the real number 5 is also a complex number, because \( 5 = 5 + 0i \). In fact, any real number is a complex number too. Do you understand the verbal irony from earlier?

Definition 3.3. Let \( z \) be the complex number such that \( z = a + bi \) where \( a, b \) are real numbers.

The real part of \( z \) is

\[
\text{Re}(z) = a
\]

and the imaginary part of \( z \) is

\[
\text{Im}(z) = b.
\]

It should be understood too that complex numbers add, subtract, and multiply in the same way you would do with variables or irrational numbers. For example, \((1 + 2i) + (3 + 4i) = 4 + 6i\). All you need to do is group the imaginary parts together and the real parts together.

Multiplication follows the same way described in algebra using the distributive property.
Example 3.4 Simplify the following:

\[(1 + 2i) \cdot (3 + 4i).\]

Solution 3.4. We expand:

\[1 \cdot (3 + 4i) + 2i \cdot (3 + 4i) = 3 + 4i + 6i - 8\]

where we used the fact that \(2i \cdot 4i = -8\). This expression reduces to

\[-5 + 10i.\]

\[\square\]

It might be rather interesting to take note that even though each real and imaginary part in the two complex numbers were positive, their product had a negative real part. The trickier example, however, lies in division. But we will first need another definition.

Definition 3.5. The complex conjugate of the complex number \(z = a + bi\) for real and imaginary parts \(a, b\) is the number

\[\bar{z} = a - bi.\]

This definition actually allows us to write expressions for the real and imaginary parts:

\[\text{Re}(z) = (z + \bar{z})/2\] and \(\text{Im}(z) = (z - \bar{z})/2.\) But the actual interesting property with the conjugate lies with the fact that multiplying a complex number \(z\) with its conjugate \(\bar{z}\) gives a real number. This actually has its own name.

Definition 3.6. The modulus of the complex number \(z\) is the real number

\[|z| = \sqrt{z \cdot \bar{z}}.\]

This is also sometimes referred to as the magnitude or the norm of a complex number. We can see that this is a real number through simple algebra: if \(z = a + bi\), then

\[|z|^2 = (a + bi)(a - bi) = a^2 + b^2.\] Let’s see how this can manage division.

Example 3.7 Simplify the following:

\[
\frac{1 + 2i}{3 + 4i}.
\]

Solution 3.7. We multiply the numerator and denominator by the conjugate of the denominator:

\[
\frac{1 + 2i}{3 + 4i} = \frac{(1 + 2i)(3 - 4i)}{(3 + 4i)(3 - 4i)} = \frac{11 + 2i}{25}.
\]
It’s usually a common theme in mathematics that division is the more difficult one out of multiplication and division.

§3.2 Complex Geometry

You might be familiar with the Cartesian plane, which is the typical set of x-y axes you learn in high school to draw graphs of functions and curves in. For instance, you would see the function $y = x^2$ being graphed as an upright parabola in the Cartesian plane. The point $(1, 2)$ would be located one unit to the right and two units up from the origin, which is $(0, 0)$.

The complex plane is quite similar. It has two axes, the real and the imaginary, which are horizontal and vertical, respectively. The origin is located at $0 + 0i$ and is typically just written as 0. The number $a + bi$ is represented by a point $a$ units to the right and $b$ units up from the origin. This brings a sense of geometry into complex numbers. Let us demonstrate this in a theorem.

**Theorem 3.8 (Modulus is Distance)** The distance of the complex number $z$ from another complex number $w$ is $|z - w|$.

**Proof.** Let $z = a + bi$ and $w = c + di$ for real and imaginary parts $a, c$ and $b, d$, respectively. Then, the point $z$ is $a - c$ units to the right and $b - d$ units up from the $w$, and Pythagorean theorem tells us that the straight distance (the hypotenuse) is

$$\sqrt{(a - c)^2 + (b - d)^2} = |z - w|.$$

This also means that the distance of $z$ to the origin is $|z|$. One complex function you can plot is the function $|z| = 1$, which is actually the unit circle. Do you see why?

You may also notice that adding two complex numbers does not necessarily result in a third that has a magnitude equal to the magnitude of the two previous complex numbers. This brings us to more geometry, specifically, trigonometry.

**Definition 3.9.** The angle of a complex number $z$ is defined as the angle between the line from the origin to the point $z$ and the real axis.

This is rather easy to compute for complex numbers.
Theorem 3.10 (Angle of Complex Numbers) The angle $\theta$ of complex number $z$ is

$$\theta = \tan^{-1} \left( \frac{\text{Im}(z)}{\text{Re}(z)} \right) = \tan^{-1} \left( \frac{z - \overline{z}}{z + \overline{z}} \right).$$

The proof of this is simple "SOHCAHTOA", or primarily the "TOA" portion. It’s typically more meaningful to understand the angle geometrically though. We can actually write the whole complex number in terms of its magnitude and angle.

Definition 3.11. The polar form of the complex number $z = a + bi$ with angle $\theta = \tan^{-1}(b/a)$ is

$$z = |z| \cdot \text{cis}(\theta) = |z|(\cos(\theta) + i \sin(\theta)).$$

Here, cis stands for ”cosine i sine”, which is the order we typically write the polar form in. This way, we can save some pen ink. It is worth noting that the complex number cis($\theta$) has magnitude 1. There might seem like no real purpose in designing the polar form, but it has a key property that justifies its existence.

Theorem 3.12 (Complex Angle Addition) For two angles $\alpha$ and $\beta$,

$$\text{cis}(\alpha + \beta) = \text{cis}(\alpha) \cdot \text{cis}(\beta).$$

Proof. Using angle addition identities of cosine and sine,

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta).$$

Expanding $\text{cis}(\alpha + \beta)$ using these identities will match the algebraic expansion of $\text{cis}(\alpha) \cdot \text{cis}(\beta)$.\qed

This gains its usefulness in multiple ways. One, multiplying two complex numbers $z$ and $w$ of angles $\alpha$ and $\beta$ will result in a complex number $zw$ with angle $\alpha + \beta$. In terms of geometry, if I wanted to rotate a point $z$ by an angle of $\alpha$, I would simply multiply $z$ by the complex number cis($\alpha$) to calculate the resulting rotated point. This has quite the niche in geometry, and many geometric problems can be transcribed into the complex plane and solved.

Another incredible property of cis is its exponent abilities.

Theorem 3.13 (De Moivre’s Theorem) For an angle $\theta$ and a positive integer $n$,

$$(\text{cis}(\theta))^n = n \cdot \text{cis}(\theta).$$
The proof can derive from a theorem which will be stated in the next subsection, or with induction, which will be covered in a future handout.

De Moivre’s Theorem has incredibly problem solving power, as test writers know this theorem and center problems specifically around it when they can’t think of harder problems to write.

Example 3.14 (AMC 12 20...) blah blah

Solution 3.14 Trivial by Shankar’s Favorite Factoring Trick. □

There are many more problems which can be solved similarly; they will be included in the problems section at the end of the handout.

§3.3 Exponential Form and Roots of Unity

Here’s a Dragon Ball Z meme.

\[ z = a + bi \]

What you’re seeing now is my normal state.

\[ z = |z|(\cos(\theta) + i\sin(\theta)) \]

This is a super saiyan.

And this...

\[ z = |z|\text{cis}(\theta) \]

... this is what is known as a super saiyan that has ascended past a super saiyan.

Or, you could just call this a super saiyan 2.

Young Mathematician: What a useless transformation. You’ve changed your hair; so what?

     Just wait.

     And this... is... to go... even further beyond!

Theorem 3.15 (Euler’s Identity) Euler states that

\[ e^{i\pi} + 1 = 0. \]
This implies, in radians for angles, that \( e^{i\pi} = -1 = \text{cis}(\pi) \). We can take this further, by stating that:

**Definition 3.16.** The **exponential form** of the complex number \( z \) is

\[
z = |z|e^{i\theta}.
\]

This may be quite a fast one, so I’ll give some further explanation. It turns out that Euler’s Identity is true (typically proven using calculus), implying that our good friend \( \text{cis} \) can actually be written in terms of an exponent. You can verify that angle addition and De Moivre’s theorems still hold with exponents; in fact, they’re incredibly straightforward as results of exponent properties such as \( e^a \cdot e^b = e^{a+b} \) and \( (e^a)^b = e^{ab} \). We will still occasionally refer to De Moivre’s theorem by its rightful title, but under non-ambiguous circumstances, we may omit the name and simply refer to the property of exponents.
§4 Polynomials

§4.1 The basics

A polynomial is the sum of one or more terms, each of which are a different power of some variable (that is an integer) multiplied by a number. For example, $2x + 1.3, 1, x^5 + x^4 + x^3 + 2x + \pi$, and $x^{77878} + (2 + 3i)x$ are polynomials, but $2^x + 4x$ is not.

**Definition 4.1.** A real polynomial is a polynomial whose coefficients are all real numbers.

For example, our earlier example of $x^{77878} + (2 + 3i)x$ is not a real polynomial because the coefficient of $x$ (which is $2 + 3i$) is not a real number.

**Definition 4.2.** A integer polynomial is a real polynomial whose coefficients are all integers.

For example, out earlier example of $x^5 + x^4 + x^3 + 2x + \pi$ is not an integer polynomial because the coefficient of $x^0$ (which is $\pi$) is not an integer.

Polynomials represent a significant part of Algebra. In particular, finding the solutions to polynomial equations is something that mathematicians have been striving to do for thousands of year.

For quadratic polynomials- polynomials with degree of two (the degree is the largest exponent of any term of the polynomial)- a solution has been found. This is the following theorem:

**Theorem 4.3** *(Quadratic Formula)* The two solutions to the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

**Proof.** We divide both sides of the equation by $a$ to get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

We complete the square to get

$$(x + a)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$  

Simplifying, we get the desired result. 

§4.2 Specific polynomial factorizations

In this section we will cover some specific polynomial factorizations. We begin with the following simple ones:
Lemma 4.4 (Some Basic Factorizations)

\[ x^2 - y^2 = (x + y)(x - y) \]
\[ x^3 + y^3 = (x + y)(x^2 - xy + y^2) \]
\[ x^3 - y^3 = (x - y)(x^2 + xy + y^2) \]

These are easy enough to prove; just expand the right-hand side of each of them and verify that it matches the left-hand side. Now we cover a more advanced one.

Theorem 4.5 (Sophie-Germain Identity)

\[ a^4 + 4b^4 = ((a + b)^2 + b^2)((a - b)^2 + b^2) \]

Proof.

\[ a^4 + 4b^4 = a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2 = (a^2 + 2b^2)^2 - (2ab)^2 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab). \]

Finally, we cover some more complex factorizations.

Theorem 4.6 (More Complex Factorizations)

\[ x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1) \]
\[ a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \]

§4.3 Example Problems

Example 4.7 (1987 AIME) Compute

\[
\]

Solution 4.7. (AoPS) The Sophie Germain Identity states that \( a^4 + 4b^4 \) can be factorized as \( (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab) \). Each of the terms is in the form of \( x^4 + 324 \). Using Sophie-Germain, we get that

\[ x^4 + 4 \cdot 3^4 = (x^2 + 2 \cdot 3^2 - 2 \cdot 3 \cdot x)(x^2 + 2 \cdot 3^2 + 2 \cdot 3 \cdot x) = (x(x - 6) + 18)(x(x + 6) + 18) \]
which then equals

\[
\frac{(10(4) + 18)(10(16) + 18)(22(16) + 18)(22(28) + 18) \cdots (58(52) + 18)(58(64) + 18)}{(4(-2) + 18)(4(10) + 18)(16(10) + 18)(16(22) + 18) \cdots (52(46) + 18)(52(58) + 18)}
\]

Almost all of the terms cancel out! We are left with \(\frac{58(64)+18}{4(-2)+18} = \frac{3730}{16} = 233\).

**Example 4.8** (IMO Math) A monic polynomial \(f(x)\) satisfies \(f(1) = 10, f(2) = 20, f(3) = 30\). Determine \(f(12) + f(-8)\).

**Solution 4.8** Note that \(f(x) - 10x = 0\) at \(x = 1, 2, 3\); therefore, \((x - 1)(x - 2)(x - 3)|f(x) - 10x\). Thus, we can write

\[f(x) = (x - 1)(x - 2)(x - 3)(x - c) + 10x\]

knowing that \(f\) is monic. Now we see that

\[f(12) + f(-8) = (11)(10)(9)(-12-c)+10(12)+(-9)(-10)(-11)(-8-c)+10(-8) = 19840\]

**Example 4.9** (2014 USAMO) Let \(a, b, c, d\) be real numbers such that \(b - d \geq 5\) and all zeros \(x_1, x_2, x_3, x_4\) of the polynomial \(P(x) = x^4 + ax^3 + bx^2 + cx + d\) are real. Find the smallest value the product \((x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)\) can take.

**Solution 4.9** (Evan Chen)

\[
\prod (x_i^2 + 1) = P(i)P(-i) = (b - d - 1)^2 + (c - a)^2 \geq (5 - 1)^2 + 0^2 = 16.
\]